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On the development of singularities in linear dispersive systems

By ALEX D. D. CRAIK

School of Mathematics and Statistics, University of St Andrews, St Andrews, Fife KY16 9SS, Scotland, UK
addc@st-and.ac.uk

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Benilov, O'Brien & Sazonov (2003) and Benilov (2004) describe "a new type of instability" in a liquid film inside a rotating cylinder. Though their linear systems support only neutrally stable modes, they find explosive disturbances which become singular after a finite time. They suggest that this result casts doubt on the reliability of modal analysis for prediction of instability; and they claim that "Such cases have never been described in the literature, and they are probably extremely rare". Here, other examples are given, some of which have been known (though not well-known) for many years. A common feature of all these singularities is a local phase synchronization of short-wave modes; but the configuration of Benilov *et al.* has the additional feature of eigenfunctions that exhibit very large changes in amplitude within the spatial domain. The relevance, or not, of such singularities to real physical systems is discussed.

1. Fourier series and singularities

The configuration of Benilov, O'Brien & Sazonov (2003, henceforth referred to as B), led, after several assumptions which do not at present concern us, to an eigenvalue problem on a finite (periodic) interval, $0 \leqslant \theta \leqslant 2\pi$, for temporally periodic modes of exponential form $\exp(-i\omega_n t)$ and eigenfunctions $\phi_n(\theta)$, where $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. For each n, the eigenvalues ω_n are real and so each mode is neutrally stable. Accordingly, an arbitrary initial state represented as

$$h(\theta, 0) = \sum_{n=-\infty}^{\infty} a_n \phi_n(\theta) \tag{1.1}$$

has the later form

$$h(\theta, t) = \sum_{n=-\infty}^{\infty} a_n \phi_n(\theta) \exp(-i\omega_n t) \tag{1.2}$$

for times $t > 0$. It turns out that there are initial states, characterized by the Fourier coefficients a_n, for which the series converges for some initial time interval $0 \leqslant t < t_0$, but which diverges at the finite time t_0. Here, we examine why this can be so.

First we look at situations with sinusoidal eigenfunctions, thereby avoiding, for the moment, the more complicated system of B. Choosing x as the spatial variable, we suppose that a Fourier synthesis of linear modes corresponding to (1.2) gives

$$f(x,t) = \sum_{n=-\infty}^{\infty} a_n \exp(\mathrm{i}k_n x - \mathrm{i}\omega_n t)$$

for the physical variable of interest. On the interval $-l \leqslant x \leqslant l$ and with periodic boundary conditions, $k_n = \pi n/l$ and corresponding eigenvalues yield the dispersion relation $\omega_n = \Omega(k_n)$. (For simplicity, we assume that there is just one eigenvalue ω_n of physical interest for each k_n: but this is not usually so in systems dependent on more than one spatial variable.)

We are mostly concerned with Fourier modes that are either neutrally stable (ω_n real) or damped ($\mathrm{Im}\omega_n < 0$). But we first remark that known cases with temporally unstable modes ($\mathrm{Im}\omega_n > 0$, some n) exhibit finite-time singularities, even though individual modes grow only exponentially in time: see, for example, Jones & Morgan (1972), Jones (1973), Saffman & Baker (1979) and Craik (1983) on the linear growth of unstable disturbances on a vortex sheet. Thus, for an unbounded inviscid incompressible flow, with velocity profile $u = V \mathrm{sgn} z$ in the x-direction with z measured transversely, an initial disturbance with Fourier transform $F(\alpha)$ is

$$f(x,0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \exp(\mathrm{i}\alpha x)\, \mathrm{d}\alpha. \tag{1.3}$$

At later times t, this turns out to become (see Craik 1983, p. 86)

$$f(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \cosh(\alpha V t) \exp(\mathrm{i}\alpha x)\, \mathrm{d}\alpha, \tag{1.4}$$

and this integral converges for all x at time t if and only if

$$\lim_{\alpha \to \pm\infty} \{F(\alpha) \exp|\alpha V t|\} = 0.$$

In this case, initial disturbances remain bounded at all later times t only if $|F(\alpha)|$ decays more rapidly than any exponential $\exp(-K|\alpha|)$ ($K > 0$) as $|\alpha|$ approaches infinity: such an example with $F(\alpha)$ proportional to $\exp(-K\alpha^2)$ is given by Drazin & Reid (1981, p. 29). But disturbances with $F(\alpha)$ proportional to $\exp(-K|\alpha|)$ (some $K > 0$) as $|\alpha|$ approaches infinity will become singular at the finite time $t_0 = K/|V|$.

Correspondingly, any bounded initial disturbance with $F(\alpha)$ decaying asymptotically to zero *more slowly than any exponential* $\exp(-K|\alpha|)$ (all $K > 0$) must become singular at *all* instants $t > 0$, however small (see Craik 1983, p. 87). When $F(\alpha)$ is wholly real, this singularity, if it occurs, must do so at the point $x = 0$, the disturbance remaining finite at all other points x due to phase mixing. Such singularity formation due to rapid growth of short-wave components turns out to be connected with the singularities discussed in §§ 3 and 4 below.

The most familiar type of singularity is the *Dirac delta function* $\delta(x - x_0)$: this generalized function equals zero for all $x \neq x_0$, and its integral from a to b ($>a$) equals unity whenever $a < x_0 < b$. The Fourier series representation of the Dirac delta function $\delta(x)$ is

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(\mathrm{i}\alpha x)\, \mathrm{d}\alpha.$$

Similarly, an infinite row of equally spaced delta functions at $x = 2nl$ $(n = 0, \pm 1, \pm 2, \pm 3, \ldots)$ has the Fourier series (see Lighthill 1962, pp. 19, 68)

$$\sum_{n=-\infty}^{\infty} \delta(x - 2nl) = \frac{1}{2l} \sum_{n=-\infty}^{\infty} \exp(\mathrm{i} n\pi x/l). \tag{1.5}$$

The singular solution (4.5) in B is of delta-function type (see equation (3.7) below), as are those discussed in this paper. (Higher-order singularities associated with other generalized functions might be studied similarly, but we do not do so.)

2. Examples with neutral sinusoidal modes

We consider disturbances of the form

$$f(x,t) = \frac{1}{\pi}\int_0^{\infty} \cos(\alpha x)\cos(\omega t)\,\mathrm{d}\alpha \quad \text{or} \quad \frac{1}{\pi}\int_0^{\infty} \cos(\alpha x - \omega t)\,\mathrm{d}\alpha. \qquad (2.1\ a, b)$$

where $\omega = \omega(\alpha)$ is some known positive real function of the wavenumber α. Both forms have a delta-function singularity at $x = 0$ and $t = 0$. Since this $f(x,t)$ has dimensions of $(\text{length})^{-1}$, a corresponding displacement is obtained by multiplying this expression by the square of some chosen length scale. The form (2.1*a*) comprises symmetric standing-wave modes, and (2.1*b*) contains only wave modes that propagate from left to right. We are interested in whether these expressions do or do not have a singularity at other times t.

(*a*) *Inviscid gravity waves.*

For water of infinite depth, Lamb (1945, pp. 384–394) considers the form (2.1*a*) with $\omega = (g\alpha)^{1/2}$, where g is gravitational acceleration. He finds that

$$f(x,t) = \frac{1}{\pi x}\left[\zeta - \frac{\zeta^3}{3.5} + \frac{\zeta^5}{3.5.7.9} - \ldots + \frac{\zeta^{2N+1}(-2)^N(2N)!}{(4N+1)!} + \ldots\right], \quad \zeta = \frac{gt^2}{2x}. \tag{2.2}$$

The function in square brackets was tabulated by Lommel (1886), and Lamb gives diagrams showing $f(x,t)$ versus x at fixed times t, and versus t at fixed locations x. Lamb's derivation draws on the early work of Cauchy (1827) and Poisson (1818) on the solution of the general initial-value problem for water waves, and on work of Lommel and Fresnel on diffraction. (See the historical accounts of Darrigol 2003 and Craik 2004.) For large ζ, a more convenient asymptotic approximation was given by Poisson (1818) (see Lamb 1945, p. 387):

$$f(x,t) = x^{-1}(\zeta/2\pi)^{1/2}\cos\left(\frac{1}{2}\zeta - \frac{1}{4}\pi\right) - (\pi x)^{-1}[\zeta^{-1} - 3.5\zeta^{-3} + 3.5.7.9\zeta^{-5} - \ldots].$$

Waves spread out symmetrically from the origin, longer waves travelling faster and farther than shorter ones, as expected from the dispersion law. At each positive time t, the amplitudes of wave crests diminish and the local wavelength increases with distance. More importantly for our purposes, a singularity remains at $x = 0$ for *all subsequent times t*. Regarding this singularity, Lamb observed (p. 392) that:

> the region in the immediate neighbourhood of the origin may be regarded as a kind of source, emitting … an endless succession of waves …. This persistent activity of the source is not paradoxical; for our assumed initial accumulation of a finite volume of elevated fluid on an infinitely narrow base implies an unlimited store of energy.
>
> In any practical case, however, the initial elevation is distributed over a band [of wavenumbers] of finite breadth … In the result, the mathematical infinity and other perplexing peculiarities … disappear.

Since (2.2) is a solution for $t < 0$ as well as $t > 0$, it follows that all disturbances leading to a delta-function singularity at some later time are themselves singular at $x = 0$ at all previous times t. Therefore, in contrast with the model of B, *for gravity waves in deep water, no bounded initial state can evolve into a delta-function singularity at some later time.*

(*b*) *Case* $\omega(\alpha) = \alpha + \varepsilon^2\alpha^3$.

This is a simplification of the dispersion relation that arose in B (see their p. 208). Though our configuration is simpler than theirs, needing only sinusoidal eigenfunctions, the evolution of the travelling-wave configuration (2.1*b*) is in many respects analogous. With the change to a moving coordinate $\xi = x - t$, we have

$$f(x,t) = \frac{1}{\pi}\int_0^\infty \cos(\alpha\xi - \varepsilon^2 t\alpha^3)\,\mathrm{d}\alpha = \frac{1}{(3\varepsilon^2 t)^{1/3}}\mathrm{Ai}\left[\frac{-\xi}{(3\varepsilon^2 t)^{1/3}}\right] \quad (t > 0),$$

$$= \frac{1}{(-3\varepsilon^2 t)^{1/3}}\mathrm{Ai}\left[\frac{\xi}{(-3\varepsilon^2 t)^{1/3}}\right] \quad (t < 0).$$

Here $\mathrm{Ai}(-u)$ is the Airy function (see e.g. Abramowitz & Stegun 1965, pp. 446–447), which has the form of slowly decreasing waves as u increases from zero, and uniform decay to zero as u takes increasingly large negative values. Here, $f(\xi, t)$ is singular only at $(\xi, t) = (0, 0)$: at all earlier and later times t, the surface elevation remains bounded for all ξ. Hence, an initial bounded state, corresponding to the above solution at any specified negative time $t = -t_0$, will become singular at the origin $x = 0$ after a time t_0 has elapsed. Likewise, an initial delta-function singularity becomes non-singular at all later times. This is unlike case (*a*), where the singularity persists, and it is analogous to the solutions found by B.

However, unlike the problem of B, the present system is conservative, with constant energy. Accordingly, if a singularity exists at a single instant and at a single point, then the total energy of the disturbance must be infinite at all previous and later times. Here, the group velocities of short wavelengths equal $3(\varepsilon\alpha)^2$, an expression which becomes indefinitely large as $|\alpha| \to \infty$. Thus the energy contained in sufficiently short waves is transported from left to right at indefinitely large speeds. It is for this reason that the singularity cannot persist for more than an instant. In contrast, the group velocity in case (*a*) decays to zero as α approaches infinity, and indefinitely short waves are unable to leave the vicinity of the origin: as a result, any singularity, if it exists, must persist for all times t.

(*c*) *Capillary waves*

Here, $\omega(\alpha) = T^{1/2}\alpha^{3/2}$ where T is the coefficient of surface tension divided by density. The resulting integrals (2.1*a*, *b*) do not then reduce to tabulated functions. However, it is easy to see that, for all non-zero t, there is no singularity at $x = 0$: this is because $\int_0^\infty \cos(b\alpha^{3/2})\mathrm{d}\alpha$ is bounded for all non-zero constants b. The dominant contribution to (2.1*a*, *b*) can be found by using Kelvin's method of stationary phase: see Kelvin (1887), Lamb (1945, pp. 395–398 and 462–463), and Copson (1971). Kelvin observed that his method may be applied equally well to gravity waves in water of finite depth, to deep-water capillary–gravity waves and to light in a dispersive medium.

Lamb states the leading-order result for an initial delta-function force impulse at the surface of still water, but he does not consider the delta-function surface displacement of present interest. Though Lamb's approximate result (1945, p. 463, equation 8) appears to indicate a singularity at $x = 0$ for all times t, this stationary-phase approximation is valid only if $T^{1/2}t/x^{3/2}$ is small, and so cannot be applied at

$x=0$. At $x=0$, Lamb's exact integral (p. 462, equation 1) has the form

$$\int_0^\infty \cos\left(b\alpha^{3/2}\right)\alpha^{-1/2}\mathrm{d}\alpha \quad \left(b=T^{1/2}\right),$$

which is clearly bounded.

The corresponding stationary-phase approximation for the delta-function displacement (2.1*b*) is

$$f(x,t)\approx\frac{4x^{1/2}}{3(\pi T)^{1/2}t}\cos\left(\frac{4x^3}{27Tt^2}-\frac{\pi}{4}\right)+\ldots \quad (x,t>0) \qquad (2.3a)$$

$$\approx\frac{4x}{3(-\pi Tx)^{1/2}t}\cos\left(\frac{4x^3}{27Tt^2}+\frac{\pi}{4}\right)+\ldots \quad (x,t<0), \qquad (2.3b)$$

which has no singularity at $x=0$. Respectively, these are good approximations for sufficiently large positive x at each constant $t>0$, and for sufficiently large negative x at each constant $t<0$; but again neither can be relied upon when x is small. However, we have just confirmed that there is no singularity at $x=0$ except when $t=0$. When x and t differ in sign, there are no equivalent results since there is then no point of stationary phase. Results (2.3*a*, *b*) respectively show that, when x or $-x$ is large for each fixed positive or negative t, both the local amplitude and wavenumber continuously increase with $|x|$. (In contrast, the Airy function in case (*b*) gives wave amplitudes that decay asymptotically as $x^{-1/4}$ for fixed t of $O(x)$.) Here, as in case (*b*), *there is a large class of disturbances, bounded for all finite x, that become singular at some later time*. But these disturbances do not tend to zero as $|x|$ approaches infinity, except at the precise moment when the singularity appears.

It seems that a graph of the solution (2.1*b*) for capillary waves has not previously been published. The integral may be rewritten as

$$f(x,t)=\frac{1}{\pi(Tt^2)^{1/3}}\int_0^\infty\cos\left(u\xi-u^{3/2}\right)\mathrm{d}u=\frac{2}{3\pi(Tt^2)^{1/3}}\int_0^\infty\cos\left(\xi v^{2/3}-v\right)\frac{\mathrm{d}v}{v^{1/3}}$$

where $\xi\equiv x(Tt^2)^{-1/3}$; but neither of these forms is convenient for computation. Contour integration of the latter around the first quadrant yields the more suitable form

$$f(x,t)=\frac{2}{3\pi(Tt^2)^{1/3}}\int_0^\infty\exp\left(-w+\frac{\sqrt{3}}{2}\xi w^{2/3}\right)\cos\left(\frac{1}{2}\xi w^{2/3}-\frac{\pi}{3}\right)\frac{\mathrm{d}w}{w^{1/3}}$$

$$\equiv\frac{2}{3\pi(Tt^2)^{1/3}}F(\xi) \qquad (2.4)$$

in which the integrand with respect to w decays exponentially at large w. The function $F(\xi)$ was calculated using MAPLE for $-3<\xi<7$: this is shown as the solid curve in figure 1. Also shown, as a dashed curve, is the stationary-phase approximation to $F(\xi)$, corresponding to result (2.3*a*), which is

$$2(\pi\xi)^{1/2}\cos\left[\tfrac{4}{27}\xi^3-\tfrac{1}{4}\pi\right] \quad (\xi>0). \qquad (2.5)$$

Agreement between $F(\xi)$ and its stationary-phase approximation is very good for ξ greater than 3: the actual error is plotted in figure 2. Still better agreement could be obtained by calculating higher-order asymptotic approximations. The author is most grateful to Peter Lindsay for undertaking these computations.

The analytical results described in cases (*a*) and (*c*) were derived in the 19th century. But these earlier researches envisaged the subsequent evolution of an initial

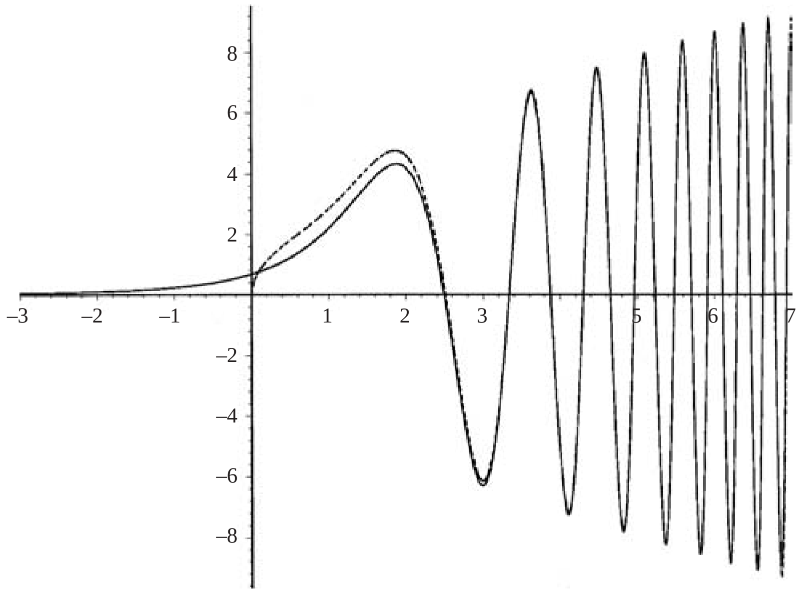

FIGURE 1. The function $F(\xi)$ defined in (2.4) (solid line) and its stationary-phase approximation (2.5) (dashed line).

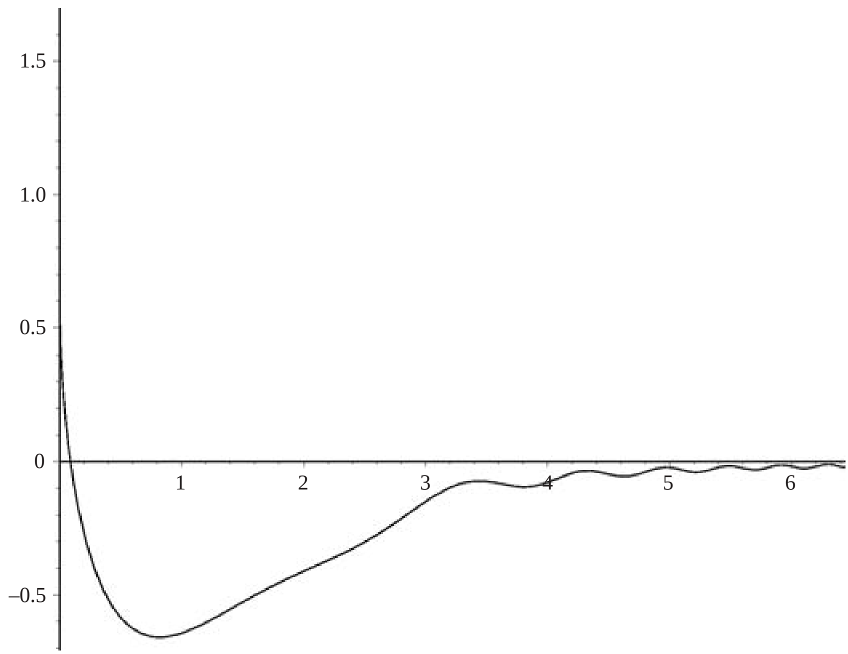

FIGURE 2. The difference between $F(\xi)$ and the approximation (2.5).

localized pulse, rather than the later development of a singularity from smooth initial data. Though the time-reversibility of the conservative equations of motion was well understood, it seems that 19th-century hydrodynamicists chose to ignore the latter consequence of their work. Aware of the many paradoxical and unrealistic results deriving from the equations of inviscid hydrodynamics, they were perhaps reluctant to draw attention to yet more.

(*d*) *General non-dissipative case*

The above examples show that the persistence, or not, of an initial delta-function singularity is determined by the nature of the dispersion relation at large wavenumbers α. If ω behaves asymptotically as $K\alpha^N$ for large α, where K and N are constants, then the singularity at $x = 0$ persists for all t provided N is less than 1, and the singularity disappears for all non-zero t whenever N is greater than 1. Similarly, if $\omega = K\alpha + L\alpha^M + \ldots$ when α is large and $M < 1$, singularities persist for all t, now located at $x = \pm Kt$ for (2.1*a*) and $x = Kt$ for (2.1*b*); but these disappear for all non-zero t if $M > 1$.

Accordingly, in case (*a*), no bounded state for $t < 0$ can lead to a delta function at $t = 0$ because $N = 1/2 < 1$; hence no such singularity can develop at any later time from any bounded initial data. But, in (*b*) and (*c*), there are classes of initial states (bounded except perhaps as $|x|$ approaches infinity), that develop delta-function singularities at some later time: in (*b*), $N = 1$, $M = 3 > 1$; and in (*c*), $N = 3/2 > 1$.

It should not be supposed that only a very limited class of initial disturbances gives rise to such singularities. We have described only those solutions that evolve into a pure delta function; but to these one may add any combination of modes with finite wavenumbers. The appearance of a singularity depends only on the synchronization, at some (x, t), of the phases of infinitely short wavelengths, and (in the present conservative situations) on infinite energy being contained therein.

A general disturbance may be expressed as the Fourier integral

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\alpha)\exp[\mathrm{i}(\alpha x - \omega t]\,\mathrm{d}\alpha.$$

Singularities like those above occur at $x = 0, t = 0$ whenever $F(\alpha)$ approaches a constant as $|\alpha|$ tends to infinity. If, instead, $F(\alpha) \to M\exp(\mathrm{i}p\alpha)$ as $|\alpha| \to \infty$ where M and p are real constants, the singularity at $t = 0$ is merely shifted to $x = -p$. Likewise, if ω behaves asymptotically as $K\alpha^N$ where $N > 1$, and $F(\alpha) \to M\exp(\mathrm{i}p\alpha + \mathrm{i}q\alpha^N)$, the singularity exists only at $x = -p$, $t = q/K$. On the other hand, if $F(\alpha) \to M\alpha^r$ where M and $r > 0$ are real constants, more-exotic higher-order singularities arise.

Although B (p. 211) claim that such "exploding" solutions "have never been described in the literature, and they are probably extremely rare", it is clear from the above that such solutions are common in non-dissipative dispersive systems for which the frequency at large wavenumbers increases as some power of α that is greater than 1. All that is required is that short waves remain dispersive, with group velocities that do not tend to a constant as α approaches infinity. However, the system of B is not a conservative one, and their geometry less simple, with eigenmodes that are not just plane waves. Though one may expect the behaviour of such systems to show qualitative similarities with those just described, it is necessary now to examine carefully the roles played by dissipation and by other geometries.

3. The problem of Benilov *et al.*

Other geometries may not support plane waves, but may allow decomposition into a complete set of neutrally stable modes of form $\phi_n(\theta)\exp(-\mathrm{i}\omega_n t)$ $(n = 1, 2, 3, \ldots)$ where θ is some spatial variable, and the real mode frequencies ω_n and usually complex eigenfunctions $\phi_n(\theta)$ are known. (In B's case, θ is an angle with range $0 \leqslant \theta \leqslant 2\pi$ and there are periodic boundary conditions.) If extended to infinite regions, as in B's local WKB approximation, summation over n is replaced by integration over a continuum of modes, as

$$\int_{-\infty}^{\infty} F(\alpha)\phi(\alpha,\theta)\exp(-\mathrm{i}\omega t)\,\mathrm{d}\alpha,$$

where ω is a known function of α. Often, the higher eigenfunctions for large $|\alpha|$ are nearly sinusoidal, at least locally in θ. On the infinite periodic extension of finite domains, one has not a single delta function but a periodic array of them, with spacing equal to that of the domain, rather as in (1.3). The development or annihilation of a singularity is still entirely dominated by the behaviour of the higher eigenmodes, as n or α approaches infinity. We expect a singularity to arise from suitable bounded initial data if the dispersion relation for ω is such that these higher modes coalesce in phase at some (θ, t), rather than phase mix as they do at others. Likewise, initial singularities should disappear for $t > 0$ if and only if ω_n or $\omega(\alpha)$ grow rapidly enough as n or $|\alpha|$ increases towards infinity.

In the problem of B, the eigenfunctions $\phi_n(\theta)$ satisfy the non-self-adjoint equation (B, equation (3.2))

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\varepsilon\sin\theta\frac{\mathrm{d}\phi}{\mathrm{d}\theta}\right)+\frac{\mathrm{d}\phi}{\mathrm{d}\theta}-\mathrm{i}\omega\phi=0 \tag{3.1}$$

where $\phi=\phi_n(\theta)$ and $\omega=\omega_n$ $(n=1,2,3,\ldots)$ are the set of complex eigenfunctions and real eigenvalues associated with the periodic boundary conditions $\phi(0)=\phi(2\pi)$ and $\mathrm{d}\phi/\mathrm{d}\theta(0)=\mathrm{d}\phi/\mathrm{d}\theta(2\pi)$.

The differential equation adjoint to (3.1) is

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\varepsilon\sin\theta\frac{\mathrm{d}\psi}{\mathrm{d}\theta}\right)-\frac{\mathrm{d}\psi}{\mathrm{d}\theta}-\mathrm{i}\omega\psi=0 \tag{3.2}$$

where ψ denotes the respective adjoint eigenfunctions $\psi_n(\theta)$ for each $\omega=\omega_n$, that satisfy the same periodic boundary conditions as the $\phi_n(\theta)$. It is easy to show that

$$\int_0^{2\pi}\phi_m\psi_n\mathrm{d}\theta=0\quad(m\neq n),\qquad\int_0^{2\pi}\phi_m\psi_n^{*}\mathrm{d}\theta=0\quad(\textit{all } m,\, n),$$

where * denotes complex conjugate. Furthermore, (3.2) is such that $\psi*(\theta)=K\phi(\theta+\pi)$ for some constant K. Without loss, we may impose the normalizations $K=1$ and

$$\int_0^{2\pi}\phi_n\psi_n\mathrm{d}\theta=1\quad(\textit{all } n).$$

Expressed in terms of temporally harmonic modes, B's surface displacement $h(\theta,t)$ is (in complex form)

$$h(\theta,t)=\sum_1^{\infty}a_n\phi_n(\theta)\mathrm{e}^{-\mathrm{i}\omega_n t},$$

with arbitrary complex coefficients a_n. It readily follows that there are two conserved quantities, not noted by B,

$$\int_0^{2\pi}h(\theta,t)h(\theta+\pi,t)\,\mathrm{d}\theta=0,\qquad\int_0^{2\pi}h(\theta,t)h^{*}(\theta+\pi,t)\,\mathrm{d}\theta=\sum_1^{\infty}|a_n|^2, \tag{3.3a,b}$$

where $h(\theta+\pi, t)$ is to be interpreted modulo 2π in $0 \leqslant \theta \leqslant 2\pi$. These may be rewritten in terms of real and imaginary parts of $h(\theta, t) = h_r(\theta, t) + ih_i(\theta, t)$ as

$$\int_0^{2\pi} h_r(\theta, t) h_r(\theta+\pi, t)\, d\theta = \int_0^{2\pi} h_i(\theta, t) h_i(\theta+\pi, t)\, d\theta = \frac{1}{2}\sum_1^{\infty} |a_n|^2,$$

$$\int_0^{2\pi} h_r(\theta, t) h_i(\theta+\pi, t)\, d\theta = \int_0^{2\pi} h_i(\theta, t) h_r(\theta+\pi, t)\, d\theta = 0.$$

It might be thought that these conservation laws would require $h(\theta, t)$ to remain bounded if it were so at some initial time; but this is not necessarily so, as B's "exploding" solution confirms. The reason is that $h_r(\theta, t)$ and $h_i(\theta, t)$ can become very large while $h_r(\theta+\pi, t)$, and $h_i(\theta+\pi, t)$ remain small. The approximate localized solution for $\phi(\theta)$ in equation (3.9) of B indicates how this occurs. Though restricted to eigenvalues ω such that $\varepsilon\omega$ is small, where ε is their small gravitational parameter, this solution gives

$$\phi(\theta) \approx \frac{e^{i\omega\theta} \exp[\varepsilon\omega^2(1-\cos\theta)]}{(1-4i\varepsilon\omega\sin\theta)^{1/4}}(1+O(\varepsilon^2)). \tag{3.4}$$

This shows, for instance, that $|\phi(0)|$ and $|\phi(\pi)|$ differ by the exponential factor $\exp(2\varepsilon\omega^2)$, which is large when ω is large. The corresponding $h(\theta, t)$ is

$$h(\theta, t) \approx \sum_1^{\infty} a_n \frac{e^{i\omega_n(\theta-t)} \exp\left[\varepsilon\omega_n^2(1-\cos\theta)\right]}{(1-4i\varepsilon\omega_n\sin\theta)^{1/4}}. \tag{3.5}$$

If all the constants a_n have the same fixed phase when n is large, then all these components are (nearly) in phase when $\theta = t$. A disturbance may be bounded at $t = 0$, when phase synchronization occurs near $\theta = 0$; but the θ-dependent exponential terms permit growth at later times, when synchronization of modes occurs at larger θ-values. This result suggests that blow-up first occurs at some time in the interval $(0, \pi)$ if the constants a_n are such that the expressions $|a_n|\exp\left(2\varepsilon\omega_n^2\right)$ are $O(1)$ or more relative to n, as n (and ω_n) approaches infinity. Though this criterion is unlikely to be precise, since the above $h(\theta, t)$ is an approximation that does not remain valid at very large n, it is certainly suggestive of the true situation. The simpler model of Benilov (2004) displays similar features without need for a localized approximation: see his equation (3.4).

There are similarities between these situations and that described in equation (1.3) for temporally amplified modes which reach a threshold amplitude for singularity formation at some finite time. But now it is the spatial eigenvalue structure itself that provides the amplification. The analogy becomes clearer if high-order modes are represented by a local WKB approximation, as waves with amplitudes and wavelengths that vary gradually with the moving coordinate $y \equiv \theta - t$. Then, as shown in B (equation (4.4)), one may write $h(\theta, t) \equiv H(x, t)$ where $x \equiv \varepsilon^{-1/2} y$, and $H(x, t)$ approximately satisfies the equation

$$\frac{\partial H}{\partial t} = -\sin t \frac{\partial^2 H}{\partial x^2}, \tag{3.6}$$

now regarded as having $-\infty < x < \infty$. This has particular solutions (B, equation (4.5))

$$H(x, t) = \frac{P}{\sqrt{Q - 4\sin^2 \frac{1}{2}t}} \exp\left\{-\frac{x^2}{2\left[Q - 4\sin^2 \frac{1}{2}t\right]}\right\} \tag{3.7}$$

for arbitrary constants P, Q; and these develop delta-function singularities at the time $t_c \equiv \pm 2[\arcsin(Q/4)]^{1/2}$ whenever $|Q| \leqslant 4$.

Equation (3.6) can also be solved as a superposition of plane-wave solutions of form $f(t)\exp(\mathrm{i}\alpha x)$, where $f(t) = \exp(-\alpha^2 \cos t)$. Thus, in general,

$$h(\theta, t) \equiv H(x, t) = \tfrac{1}{2\pi} \int_{-\infty}^{\infty} F(\alpha) \exp(-\alpha^2 \cos t) \exp(\mathrm{i}\alpha x)\, \mathrm{d}\alpha, \qquad (3.8)$$

which has a form reminiscent of (1.3) above, and also of Benilov (2004, equation (3.4)). To recover the solutions (3.7) one needs to set

$$F(\alpha) = (2\pi)^{1/2} P \exp[\alpha^2(1 - (Q/2))].$$

More generally, the integral (3.8) converges for all t provided

$$\lim_{\alpha \to \pm\infty} \{F(\alpha) \exp(\alpha^2)\} = 0$$

and diverges at some t in the interval $(0, \pi]$ otherwise.

The singular solutions of B, and the rather simpler ones of Benilov (2004) have two essential features: (i) a synchronization of the phases of high-numbered eigenmodes that enables them to reinforce one another, rather than phase mix, at some instant and at some location in physical space; and (ii) a characteristic high-eigenmode structure in which the greatest local maximum in amplitude is exponentially large compared with the smallest. It is the latter property that allows disturbances to remain bounded at some locations where phase synchronization occurs, but to become singular at others. In contrast, the neutrally stable plane-wave solutions examined in § 1 do not have property (ii), and so (with appropriate short-wave amplitudes) give rise to singularities whenever phase synchronization takes place.

4. An oscillating liquid layer

4.1. *The lubrication approximation*

The problem of B concerned a thin layer of viscous fluid with a free surface, adhering to the inside of a uniformly rotating circular cylinder with horizontal axis. The azimuthal angle is θ (measured from the downwards vertical) and the reference frame $y \equiv \theta - t$ moves with the mean azimuthal speed of the cylinder. If a singularity occurs, it does so at some moment during the first revolution of the cylinder. The theoretical model of B employs the viscous-dominated lubrication approximation, whereby local changes in surface elevation result from the spatial gradient of the volume flux parallel to the wall: that is,

$$\frac{\partial h}{\partial t} = -\frac{\partial q}{\partial s}, \qquad q(s, t) \equiv \int_0^h u(n, s, t)\, \mathrm{d}n,$$

where $(s,\ n)$ are distances along and normal to the wall and u is the velocity component along the wall. To a first approximation, u equals the azimuthal velocity of the rotating cylinder; and there is a smaller correction, proportional to $\sin\theta$, due to the component of gravitational force along the wall.

Since the liquid layer experiences a periodic gravitational acceleration as it moves round with the cylinder, the problem of B has close similarities with that of a plane liquid layer oscillated sinusoidally in time normal to its surface, under zero gravity. Each period of the imposed oscillation then corresponds to one revolution of B's cylinder. The latter problem is simpler than B's and is now described.

Consider a liquid layer of mean depth d situated on a plane boundary $y = 0$ that is performing periodic normal oscillations with variable acceleration $g \sin \Omega t$. Making the lubrication approximation, we have

$$\frac{\partial \eta}{\partial t} = -\frac{\partial q}{\partial x}, \qquad q(x,t) \equiv \int_0^{d+\eta} u(y,x,t)\,\mathrm{d}y, \tag{4.1}$$

where $\eta(x, t)$ is a small normal displacement of the liquid surface, and x is measured along the wall. Also, the x-momentum equation reduces to

$$\frac{\partial^2 u}{\partial y^2} = \frac{g \sin \Omega t}{\nu} \frac{\partial \eta}{\partial x}$$

where ν is the kinematic viscosity of the liquid. Assuming $|\eta(x,t)|$ to be small compared with d, and applying the boundary conditions $u = 0$ on $y = 0$ and $\partial u/\partial y = 0$ on $y = d$, u and then q are readily found. The latter is

$$q(x,t) = -\frac{d^3 g \sin \Omega t}{3\nu} \frac{\partial \eta}{\partial x}, \tag{4.2}$$

which combines with (4.1) to yield the governing equation for $\eta(x, t)$:

$$\frac{\partial \eta}{\partial t} = \frac{d^3 g \sin \Omega t}{3\nu} \frac{\partial^2 \eta}{\partial x^2}. \tag{4.3}$$

It is no surprise that this is a diffusion equation with time-periodic diffusivity, just like (3.6) above. There is no need to describe the solutions: just as for (3.6) there are some that develop delta-function singularities during the course of one period $2\pi/\Omega$, and others that do not.

4.2. *The short-wave approximation*

To test the relevance of the lubrication approximation in the formation of singularities, we now describe a short-wave viscous approximation, since it is the short-wave components that give rise to singularities, as has already been described. But the lubrication approximation assumes that waves are long compared with the liquid depth, whereas for sufficiently short waves the opposite is the case: for these, the liquid depth might as well be taken as infinite.

The viscous damping of free-surface waves in liquid layers under the influence of constant gravity g was comprehensively discussed by Basset (1888, vol. 2, pp. 309–314). We need only consider short waves in infinite depth, for which the stream function has the form

$$\psi(x,z,t) = (A\mathrm{e}^{-\alpha z} + C\mathrm{e}^{-\kappa z})\mathrm{e}^{\mathrm{i}\alpha x + \sigma t}, \qquad \kappa \equiv (\alpha^2 + \sigma/\nu)^{1/2},$$

where $\xi \equiv (\sigma/\nu)^{1/2}$ satisfies the equation

$$(\xi^2 + 2\alpha^2)^2 - 4\alpha^3(\xi^2 + \alpha^2)^{1/2} + g\alpha/\nu^2 = 0 \tag{4.4}$$

with α assumed positive (see Basset, 1888, vol. 2, equations (21), (22), (27) with notational changes).

In the limit of very large viscosity ν or very short waves, Basset found that the smallest (least-damped) root ξ of (4.4) became zero, and he wrongly discarded this as unphysical since it gives zero growth or decay rate σ (Basset, art. 521): the other roots are then strongly damped and purely viscous in origin, unconnected with the presence of gravity, and may be neglected here. Basset's discarded root became zero because he rejected the gravitational term of (4.4) in this limit. An improved approximation, valid when $g/\alpha^3\nu^2 \ll 1$, turns out to be

$$\xi^2 = -\frac{g}{2\alpha\nu^2}, \qquad \text{i.e. } \sigma = -\frac{g}{2\alpha\nu}. \tag{4.5}$$

Consider a superposition of such waves, as in (1.3) but now damped, supposing meantime that g is a positive constant as in § 2, case (*a*). Clearly, initially bounded disturbances remain bounded at all later times. Also, an initial delta-function singularity is not suppressed at later times, because the viscous damping rate of gravity waves becomes indefinitely small as the wavenumber $|\alpha|$ approaches infinity. The corresponding result for pure capillary waves, as in § 2, case (*c*) but now damped by viscosity, is obtained on replacing g in (4.5) by $T\alpha^2$. Then, the viscous damping rate σ becomes indefinitely large as $|\alpha|$ approaches infinity: as a result, it is no longer possible for capillary waves to exhibit a finite-time singularity as they could in the inviscid case of § 2, case (*c*).

To extend result (4.5) to the oscillatory layer, we need only to replace the constant g by $g \sin \Omega t$ (provided Ω is small enough that there are no Faraday-type resonances). The instantaneous damping rate is then $\sigma = -g \sin \Omega t/2\alpha\nu$, and each mode evolves very nearly as

$$\exp\left\{ \mathrm{i}\alpha x - \int_0^t \frac{g}{2\alpha\nu} \sin \Omega t \, \mathrm{d}t \right\} = \exp\left\{ \mathrm{i}\alpha x - \frac{g(1-\cos \Omega t)}{2\Omega\alpha\nu} \right\}.$$

This is very different from the behaviour predicted by the lubrication approximation. In the latter, from (4.3), the temporal factor is $\exp[-(\alpha^2 d^3 g/3\Omega\nu)(1-\cos \Omega t)]$: as α^2 increases, this gives rise to ever larger differences in the amplitudes at, for instance, $\Omega t = 0$ and π. As we have seen, with a suitable superposition of such modes, these large differences can cause a singularity at some point in the cycle. But our present more realistic results show that this is in fact impossible: sufficiently short waves have temporal behaviour that gets ever weaker as $|\alpha|$ increases to infinity. Accordingly, no delta-function singularity can evolve from bounded initial data after a finite time, in direct contradiction to the results of lubrication theory.

A similar conclusion must hold for the cylindrical configuration of B and of Benilov (2004): their lubrication approximation breaks down for sufficiently short waves, and such waves on their moving fluid layer will exhibit temporal behaviour, relative to the moving layer, that is very similar to that just described. In particular, the strong spatial variation of the maximum local amplitudes of the high eigenfunctions is entirely suppressed.

5. Discussion

There are many real situations where locally large but finite disturbances arise from initial states having much smaller amplitudes. The phase-synchronization (or focusing) phenomenon has long been recognized as a possible cause of giant waves in the ocean. Now Benilov *et al.* have drawn attention to refinements caused by large local variations in eigenvalue stucture. They have thereby identified an interesting mechanism for disturbance growth that is unconnected with the usual exponential temporal instability of individual modes. Such behaviour may well be less rare, in theoretical models, than they suggest: an attempt has been made here to outline both the circumstances in which it may be expected to arise, and the likely limitations of such theoretical models.

B suggest that modal analysis is "not a completely reliable indicator of the stability properties of a system" (B p. 217), because a finite-time singularity appeared in their system although all modes were neutrally stable. But such anxiety seems misplaced for several reasons.

First, we found that, in conservative systems where all modes are neutrally stable, only initial disturbances with infinite energy can give rise to a singularity at later times, since the energy of the delta function is infinite. But such initial disturbances are clearly unphysical. In case (a), this infinite amount of energy resides near $x = 0$ for all t, and so the singularity cannot disappear. But, in cases (b) and (c), the infinite energy of the singularity at $(x, t) = (0, 0)$ is instantaneously transmitted to/from infinitely large distances $|x|$ at later/earlier times. However, all these scenarios are unrealistic artifacts of the non-viscous theoretical models.

Secondly, if a local singularity occurs, it does so entirely because of the contribution of modes at the high end of the wavenumber spectrum. But, in real physical systems, very short waves are certain to experience strong dissipation by viscosity, even if longer waves are neutrally stable or even unstable (cf. the quotation from Lamb given in §2 case (a) above). If, for all α greater than some large but finite constant α_0, this dissipation gives an exponential decay factor of $\exp(-K\alpha^p t)$, with positive constants K and p, then the Fourier integral corresponding to (2.1b) has a large-wavenumber contribution

$$\frac{1}{\pi}\int_{\alpha_0}^{\infty} \exp(-K\alpha^p t)\cos(\alpha x - \omega t)\,\mathrm{d}\alpha.$$

This integral is finite for all $t > 0$, whatever $\omega(\alpha)$ may be. It follows that an initial delta-function singularity at $t = 0$ yields finite displacements at all later times $t > 0$. Furthermore, since this integral diverges for all $t < 0$, no previous bounded initial state can give rise to a delta-function singularity at $t = 0$. For example, the appearance of a finite-time singularity for inviscid capillary waves is removed by dissipation, since the high-wavenumber contribution can then lose an infinite amount of energy during the first moment of time. For damped gravity waves, finite-time singularities cannot appear, but for a different reason: short gravity waves are damped as $\exp(-K\alpha^p t)$ where p $(= -1)$ is negative, and so the role of damping is ineffectual, and a singularity exists at all times or at none.

The problems of B and of Benilov (2004) are based on the lubrication approximation. Subject to the same assumptions, their analysis applies also to liquid layers on the outside, as well as the inside, of uniformly rotating cylinders. With this approximation, finite-time singularities can indeed occur, as also in the oscillating fluid layer discussed in §4.1 above. Clarification of the properties of these approximate models is a worthwhile objective, whether or not the approximations on which they are based still hold as the singularity is approached. But we showed in §4.2 that, because the lubrication approximation fails for very short waves, these singularities cannot arise: the short waves are far more strongly damped than lubrication theory suggests.

Nevertheless, Benilov *et al.* have drawn attention to an interesting mathematical phenomenon that demanded understanding. As in conservative systems, such singularities are brought about by phase synchronization of high-frequency Fourier components at some spatial location, much as in the cases (b) and (c) of §2. But, in B and Benilov (2004), the behaviour is also influenced by the unusual form of the spatial eigenfunctions, which admit very large local variations in amplitude. As a result, phase synchronization may give rise to bounded disturbances at some locations, and unbounded ones at others. If a singularity is to appear, it does so during the first 'turn' of their rotating cylinder. Similarly, for the oscillating layer of § 4.1, a singularity appears, if at all, during the first period of oscillation.

Benilov *et al.* suggest that their singularities may be associated with droplet formation on their liquid layer. This could be so, since the nonlinear process of droplet formation does not require an actual singularity in the linear approximation. But the various experiments that have been performed with viscous layers on rotating cylinders do not seem to have encountered this phenomenon, though other instabilities and nonlinear formations were observed: see, e.g. Moffatt (1977), Preziosi & Joseph (1988), and additional references in Joseph *et al.* (2003). Furthermore, other theoretical models of droplet formation, such as those of disintegrating liquid jets, do not require singularities of the kind envisaged here.

Whether or not B's mechanism is relevant to the rapid growth of real disturbances, one must interpret with caution their predicted appearance of actual singularities after a finite time. In any linear theory that predicts such singularities, the short-wave end of the spectrum of disturbances is likely to be incorrectly modelled, and the singularities spurious.

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